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On L_p-Generalization of a Theorem of Adamyan, Arov, and Kreĭn

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The paper deals with problems relating to the theory of Hankel operators. Let G be a bounded simple connected domain with the boundary Γ consisting of a closed analytic Jordan curve. Denote by $\mathscr{M}_{n,p}(G)$, $1 \leqslant p < \infty$, the class of all meromorphic functions on G that can be represented in the form $h = \beta/\alpha$, where β belongs to the Smirnov class $E_p(G)$, α is a polynomial degree at most n, $\alpha \not\equiv 0$. We obtain estimates of s-numbers of the Hankel operator A_f constructed from $f \in L_p(\Gamma)$, $1 \leqslant p < \infty$, in terms of the best approximation $\Delta_{n,p}$ of f in the space $L_p(\Gamma)$ by functions belonging to the class $\mathscr{M}_{n,p}(G)$.

Key Words: meromorphic approximation; Hankel operator; Adamyan–Arov–Kreĭn theorem, s-numbers.

1. INTRODUCTION

In recent years greater attention has been paid to the theory of Hankel operators due to its relationship with approximation theory. The origin of the corresponding approximation technique (the AAK method) can be found in the papers by Adamyan, Arov, and Kreĭn [1, 2] (see also [20]).

Originally the Adamyan–Arov–Kreĭm theorem was proved for Hankel operators on the Hardy class $H_2(G)$ in the open unit disk G with the boundary Γ.

For a continuous function f the Hankel operator A_f with the *symbol* f is defined as the multiplication operator by f followed by the operator of orthogonal projection $\mathbf{P}_-$ of $L_2(\Gamma)$ onto the orthogonal complement $H_2^{\perp}(G)$ of $H_2(\Gamma)$ in $L_2(\Gamma)$. We identify elements of $H_2(\Gamma)$ with their boundary values on Γ. We have

$$A_f\alpha = \mathbf{P}_-(\alpha f), \qquad \alpha \in H_2(G).$$

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The Adamyan–Arov–Kreĭn theorem states that the best operator norm approximation of a Hankel operator A_f constructed from f by operators of rank at most n (that is, the nth singular number $s_n(A_f)$ of A_f) is also the error $\Delta_{n,\infty}$ in best approximation of f in the space $L_\infty(\Gamma)$ by meromorphic functions that can be represented in the form $h=\beta/\alpha$, where $\beta \in H_\infty(G)$ and α is a polynomial degree at most n, $\alpha \not\equiv 0$.

The theory of Hankel operators is a valuable tool for investigating extremal problems such as the best uniform approximation of analytic functions by rational functions. In particular, with the help of the Adamyan–Arov–Kreĭn theorem, it is possible to obtain theorems connecting the behavior of best rational approximations and the properties of the function to be approximated, to investigate the degree of rational approximation for different classes of analytic functions. In this regard we mention the papers by Peller and Khrushchev [13, 14], Parfenov [12], Prokhorov [18, 19], and Prokhorov and Saff [21]. The Adamyan–Arov–Kreĭn theorem reduces the study of the rate of rational approximation for an analytic function to the study of the rate of decrease of singular numbers of the Hankel operator associated with the function to be approximated.

The main result of the present paper (Theorem 2.1) extends the Adamyan–Arov–Kreĭn theorem to the case when the symbol f of a Hankel operator is a measurable function on the boundary Γ of a simple connected domain G such that $f \in L_p(\Gamma)$, $1 \leqslant p < \infty$. In this respect we mention the papers by Le Merdy [10, 11] and by Baratchart and Seyfert [3], where generalizations of the Adamyan–Arov–Kreĭn theorem are given for the case when $2 \leqslant p < \infty$ and G is the open unit disk. In [3] the Hankel operator $A_f : H_a(G) \to H_2^\perp(G)$, $1/a+1/p=1/2$, is defined for $2 \leqslant p < \infty$ and is the composition of the operator of multiplication by f and the orthogonal projection $\mathbf{P}\_$ of $L_2(\Gamma)$ onto $H_2^\perp(G)$. In this paper we extend the definition of the Hankel operator. The Hankel operator $A_f = A_{f,a} : E_a(G) \to L_{p_1}(\Gamma)/E_{p_1}(G)$ is defined as the composition of the operator of multiplication by $f \in L_p(\Gamma)$, $1 \leqslant p < \infty$, and the canonical surjection from $L_{p_1}(\Gamma)$ onto the quotient space $L_{p_1}(\Gamma)/E_{p_1}(G)$, where $1 < a \leqslant \infty$, $1 \leqslant p_1 < \infty$, $1/a+1/p = 1/p_1$, and $E_a(G)$, $E_{p_1}(G)$ are the Smirnov classes of analytic functions on a bounded simple connected domain G with the boundary Γ.

2. THEOREM 2.1

Let G be a bounded domain on the complex plane $\mathbf{C}$ bounded by a closed analytic Jordan curve Γ. We assume that Γ is endowed with the positive orientation.

Let $L_p(\Gamma)$, $1 \leqslant p < \infty$, be the Lebesgue space of functions with integrable pth power. The norm is given by

$$\|\varphi\|_p = \left(\int_\Gamma |\varphi(\xi)|^p \, |d\xi| \right)^{1/p}.$$

$L_\infty(\Gamma)$ is the space of essentially bounded functions, with the norm

$$\|\varphi\|_\infty = \operatorname*{ess\,sup}_\Gamma |\varphi(\xi)|.$$

Denote by $E_p(G)$, $1 \leqslant p \leqslant \infty$, the Smirnov class of analytic functions on G. For $1 \leqslant p < \infty$ this class consists of the functions φ for which there is a sequence of domains G_k with rectifiable boundaries having the following properties

$$G_{k+1} \subset G_k, \qquad \bar{G}_k \subset G, \qquad \bigcup_k G_k = G$$

and

$$\sup_k \int_{\partial G_k} |\varphi(\xi)|^p \, |d\xi| < \infty.$$

$E_\infty(G)$ is the class of bounded analytic functions on G. The condition

$$\int_\Gamma \frac{\varphi(\xi)\, d\xi}{\xi - z} = 0 \qquad \text{for all} \quad z \in \bar{\mathbb{C}} \setminus \bar{G} \tag{1}$$

is necessary and sufficient for $\varphi \in L_p(\Gamma)$ to be the boundary values of a function in $E_p(G)$ (see [17] and [22] for more details about the classes $E_p(G)$).

Let $f \in L_p(\Gamma)$, $1 \leqslant p < \infty$. Suppose that a and p_1 satisfy the following conditions:

$$1 < a \leqslant \infty, \qquad 1 \leqslant p_1 < \infty, \tag{2}$$

and

$$\frac{1}{a} + \frac{1}{p} = \frac{1}{p_1}. \tag{3}$$

The Hankel operator $A_f = A_{f,a}\colon E_a(G) \to L_{p_1}(\Gamma)/E_{p_1}(G)$ is defined as the composition of the operator of multiplication by the function f and the canonical surjection $\mathbf{P}$ from $L_{p_1}(\Gamma)$ onto the quotient space $L_{p_1}(\Gamma)/E_{p_1}(G)$. For any function $\alpha \in E_a(G)$ we have $A_f \alpha = \mathbf{P}(\alpha f)$ by definition. It is not hard to see that the operator A_f is a compact operator.

Fix a nonnegative integer n. Let us introduce the s-numbers of the operator A_f. Denote by $l_n(A_f)$ the nth approximation number of the operator A_f,

$$l_n(A_f) = \inf_K \|A_f - K\|, \tag{4}$$

where the infimum is taken over the collection of all linear operators $K: E_a(G) \to L_{p_1}(\Gamma)/E_{p_1}(G)$ of rank at most n, and $\|\cdot\|$ is the norm of the corresponding linear operator. If $a = \infty$ we assume in addition that all such operators K are weak* continuous.

The nth Gelfand number $d^n(A_f)$ is given by

$$d^n(A_f) = \inf_{X_{-n}} \|A_f \mid_{X_{-n}}\|,$$

where X_{-n} is an arbitrary subspace of codimension n of $E_a(G)$.

The nth Kolmogorov number $d_n(A_f)$ of A_f is defined by

$$d_n(A_f) = \inf_{X_n} \sup_{x \in E_a(G),\, \|x\|_a = 1} \inf_{y \in X_n} \|A_f x - y\|_{L_{p_1}(\Gamma)/E_{p_1}(G)},$$

where $\inf_{X_n}$ means taking infimum for all n-dimensional subspaces X_n of $L_{p_1}(\Gamma)/E_{p_1}(G)$ (see [15, 16] for more details about s-numbers).

For any nonnegative integer n denote by $\mathcal{M}_{n,p}(G)$ the class of all meromorphic functions on G representable in the form $h = \beta/\alpha$, where $\beta \in E_p(G)$ and α is a polynomial of degree at most n, $\alpha \not\equiv 0$.

The deviation of f in $L_p(\Gamma)$ from the class $\mathcal{M}_{n,p}(G)$ will be denoted by $\Delta_{n,p}$:

$$\Delta_{n,p} = \Delta_{n,p}(f; G) = \inf_{h \in \mathcal{M}_{n,p}(G)} \|f - h\|_p. \tag{5}$$

Now we may state the main result of the paper. This result relates the s-numbers of A_f with deviations (errors) $\Delta_{n,p}$.

THEOREM 2.1. *Let G be a simple connected domain bounded by a closed analytic Jordan curve Γ and let $f \in L_p(\Gamma)$, $1 \leqslant p < \infty$. Then*

$$\Delta_{n,p} = l_n(A_f) = d^n(A_f), \qquad n = 0, 1, 2, \ldots. \tag{6}$$

Theorem 2.1 is a generalization of the Adamyan–Arov–Kreĭn theorem for the case when $f \in L_p(\Gamma)$, $1 \leqslant p < \infty$, and G is a simply connected domain. Our proof of Theorem 2.1 is based on Borsuk's theorem (see [9]) and results of Tumarkin and Khavinson [24, 25] related to approximation of a function $f \in L_p(\Gamma)$ in the space $L_p(\Gamma)$ in the class $E_p(G)$. Moreover, the arguments are based on a construction of an odd continuous mapping of the $(2n+1)$-sphere S^{2n+1} into the set of Blaschke products of degree at most n first used by Fisher and Micchelli [4].

As a consequence of Theorem 2.1 we have

COROLLARY 2.1. *Let* $1 < p < \infty$ *and* $a = 2q$, *where*

$$\frac{1}{p} + \frac{1}{q} = 1;$$

then

$$\Delta_{n,p} = l_n(A_f) = d^n(A_f) = d_n(A_f), \qquad n = 0, 1, 2, \ldots. \tag{7}$$

The paper is organized as follows. Sections 3, 4, and 5 contain required auxiliary results. In Sections 6 and 7 we study some properties of the Hankel operator. The main result is proved in Section 8. In Section 9 the proof of Corollary 2.1 is given.

3. s-NUMBERS

In this section we review some facts related to s-numbers (see, for example, [15, 16]).

Let X and Y be Banach spaces, and let A belong to the normed space $\mathscr{L}(X, Y)$ of all continuous linear operators from X to Y.

For a nonnegative integer n we denote by $l_n(A)$ the *nth approximation number* of the operator A,

$$l_n(A) = \inf_K \|A - K\|,$$

where the infimum is taken over the collection of all linear operators $K: X \to Y$ of rank at most n, and $\|\cdot\|$ is the norm of the corresponding linear operator.

The *nth Gelfand number* $d^n(A)$ is defined as

$$d^n(A) = \inf_{X_{-n}} \|A\,|_{X_{-n}}\|,$$

where X_{-n} runs over all possible subspaces of codimension n of X. We say that a subspace X_{-n} of X is of codimension n, $n \geqslant 1$, if there exists n linearly independent continuous linear functionals $l_1, \ldots, l_n$ on X for which

$$X_{-n} = \{x : x \in X, \langle x, l_i \rangle = 0, i = 1, \ldots, n\},$$

where $\langle x, l_i \rangle$ is the value of a functional l_i at an element $x \in X$.

The *nth Kolmogorov number* $d_n(A)$ of A is given by

$$d_n(A) = \inf_{X_n} \sup_{x \in X, \|x\|_X = 1} \inf_{y \in X_n} \|Ax - y\|_Y,$$

where X_n is an arbitrary n-dimensional subspace of Y.

The quantities defined above are called the *s-numbers* of the operator A. Let $\{s_n\}$, $s_n = s_n(A)$, $n = 0, 1, 2, \ldots$, be a sequence of s-numbers of the operator A. Then the sequence $\{s_n\}$, $n = 0, 1, 2, \ldots$, is nonincreasing and

$$\|A\| = s_0.$$

Moreover,

$$s_{n+m}(A+B) \leqslant s_n(A) + s_m(B)$$

and

$$|s_n(A) - s_n(B)| \leqslant \|A - B\| \tag{8}$$

for $A, B \in \mathscr{L}(X, Y)$.

It can be seen that if X and Y are Hilbert spaces and A is a compact operator, then the sequence $\{s_n\}$, $n = 0, 1, 2, \ldots$, coincides with the sequence of the singular numbers of the operator A_f. We remark that s_n is equal to the $(n+1)$st largest eigenvalue (counting multiplicity) of the operator $(A^*A)^{1/2}$, where $A^*\colon Y \to X$ is the adjoint of A. In particular,

$$l_n(A) = d^n(A) = d_n(A), \qquad n = 0, 1, 2, \ldots.$$

Let $A \in \mathscr{L}(X, Y)$. Denote by $A'\colon Y' \to X'$ the dual of A, where X' and Y' are the duals of X and Y, respectively. For an arbitrary nonnegative integer n we have

$$\max(d_n(A), d^n(A)) \leqslant l_n(A) \tag{9}$$

and

$$d^n(A) = d_n(A'). \tag{10}$$

4. AUXILIARY ASSERTIONS

Let $g(z, \xi)$ be the Green's function of the domain G with singularity at the point ξ and let $h(z, \xi)$ be the harmonic conjugate of $g(z, \xi)$. For a point $\xi \in G$ the function

$$B(z, \xi) = \exp(-(g(z, \xi) + ih(z, \xi)))$$

vanishes at ξ and is bounded by 1.

Given any points $z_1, \ldots, z_n \in G$, and $\theta \in [0, 2\pi]$, let

$$B(z) = e^{i\theta} \prod_{j=1}^{n} B(z, z_j)$$

be the Blaschke product of degree n with zeros at $z_1, \ldots, z_n$. Denote by $\mathscr{B}_n$ the set of all Blaschke products of degree at most n.

Since Γ is a closed analytic Jordan curve, the Smirnov class $E_p(G)$, $1 \leqslant p < \infty$, coincides with the Hardy class $H_p(G)$ of analytic functions φ on G (see [23, 26]). The Hardy class $H_p(G)$, $1 \leqslant p < \infty$, consists of the functions φ such that $|\varphi|^p$ has a harmonic majorant.

Questions of factorization of functions in the Hardy class $H_p(G)$ have been thoroughly investigated (see, for example, [6, 8, 27]).

Let $d\omega_z$ be the harmonic measure on Γ with respect to a point $z \in G$. We say $\varphi \in H_p(G)$ is an outer function if for all $z \in G$

$$\ln |\varphi(z)| = \int_\Gamma \ln |\varphi(\xi)| \, d\omega_z(\xi).$$

A bounded nonvanishing analytic on G function φ is a singular function if $|\varphi(\xi)| = 1$ almost everywhere on Γ.

Each function φ in $H_p(G)$ has a factorization $\varphi = BS\mathrm{M}$ where B is a Blaschke product, S is a singular function, and M is an outer function. These factors are unique up to multiplicative constants of modulus one.

5. THE PROBLEM OF THE BEST APPROXIMATION IN $L_p(\Gamma)$ BY FUNCTIONS IN $E_p(G)$

In this section we consider the case when $n = 0$. According to the definition of $\Delta_{0,p}$,

$$\Delta_{0,p} = \Delta_{0,p}(f; G) = \inf_{h \in E_p(G)} \|f - h\|_p.$$

The questions of approximation of a function $f \in L_p(\Gamma)$ in the space $L_p(\Gamma)$ in the class $E_p(G)$ were considered by Tumarkin and Khavinson (see [24, 25]; for the case when G is the unit disk see [6] and [8]).

Applying the standard arguments of dual spaces, we obtain the following lemma.

LEMMA 5.1. *Let* $f \in L_p(\Gamma)$, $1 \leqslant p < \infty$, $1/p + 1/q = 1$. *Then*

(i)

$$\Delta_{0,p} = \sup_{\tilde{\varphi} \in E_q(G),\, \|\tilde{\varphi}\|_q = 1} \left| \int_\Gamma (\tilde{\varphi} f)(\xi)\, d\xi \right|; \tag{11}$$

(ii) *there exists a unique function* $h \in E_p(G)$ *such that*

$$\Delta_{0,p} = \|f - h\|_p;$$

(iii) *there exists a unique function* $\varphi \in E_q(G)$, $\|\varphi\|_q = 1$, *on which the supremum in* (11) *is attained;*

(iv) *almost everywhere on* Γ

$$\varphi(\xi)(f-h)(\xi)\, d\xi = \Delta_{0,p}\, |\varphi(\xi)|^q\, |d\xi| \qquad if \quad 1 < p < \infty, \tag{12}$$

and

$$\varphi(\xi)(f-h)(\xi)\, d\xi = |(f-h)(\xi)|\, |d\xi| \qquad if \quad p = 1.$$

Moreover, the function $\varphi(f-h)$ *can be continued analytically across* Γ, *and* φ *has no a nonconstant singular factor if* f *is holomorphic on* Γ.

In connection with Lemma 5.1 we remark that for $1 \leqslant p < \infty$ the dual of the quotient space $L_p(\Gamma)/E_p(G)$ is $E_q(G)$, $1/p + 1/q = 1$, and $L_q(\Gamma)/E_q(G)$ is the dual of $E_p(G)$. Moreover, each continuous linear functional l on $L_p(\Gamma)/E_p(G)$, $1 \leqslant p < \infty$, can be represented in the form

$$\langle x, l \rangle = \int_\Gamma g(\xi)\, s(\xi)\, d\xi,$$

where $x \in L_p(\Gamma)/E_p(G)$ is an equivalence class $g + E_p(G)$, $g \in L_p(\Gamma)$, and $s \in E_q(G)$.

6. PROPERTIES OF THE HANKEL OPERATOR

According to the definition of the Hankel operator, A_f is the composition of the operator of multiplication by f and the canonical surjection $\mathbf{P}$ from $L_{p_1}(\Gamma)$ onto $L_{p_1}(\Gamma)/E_{p_1}(G)$. Therefore, for any function $\alpha \in E_a(G)$ we have

$$\|A_f \alpha\|_{L_{p_1}(\Gamma)/E_{p_1}(G)} = \inf_{\tilde{\beta} \in E_{p_1}(G)} \|\alpha f - \tilde{\beta}\|_{p_1}. \tag{13}$$

It follows directly from (13)

$$\|A_f \alpha\|_{L_{p_1}(\Gamma)/E_{p_1}(G)} \leqslant \|\alpha\|_a \, \|f\|_p$$

and

$$\|A_f\| \leqslant \|f\|_p, \tag{14}$$

where $\|A_f\|$ is the norm of A_f. We have the equality

$$A_{f+g} = A_f + A_g, \qquad f, g \in L_p(\Gamma). \tag{15}$$

It is not hard to see that the Hankel operator A_f is a compact operator. Indeed, let r be an arbitrary rational function with poles off Γ. By (14), (15), we get

$$\|A_f - A_r\| = \|A_{f-r}\| \leqslant \|f - r\|_p.$$

Using now the fact that A_r is an operator of finite rank and the fact that for $f \in L_p(\Gamma)$ there exists a sequence $\{r_\nu\}$, $\nu = 0, 1, 2, \ldots$, of rational functions with poles off Γ such that $\|f - r_\nu\|_p \to 0$ as $\nu \to \infty$, we can conclude that A_f is a compact operator.

7. SOME FORMULAS

Let $1 < b \leqslant \infty$ and

$$\frac{1}{p_1} + \frac{1}{b} = 1.$$

We can see immediately from the definition of the nth Gelfand number $d^n(A_f)$ that

$$\begin{aligned} d^0(A_f) = \{\sup |\langle A_f u, l \rangle| : u \in E_a(G), \|u\|_a = 1, \\ l \in (L_{p_1}(\Gamma)/E_{p_1}(G))', \|l\| = 1\} \end{aligned}$$

and

$$\begin{aligned} d^n(A_f) = \inf_{y_1, \ldots, y_n} \{\sup |\langle A_f u, l \rangle| : u \in E_a(G), \|u\|_a = 1, \\ y_i(u) = 0, i = 1, \ldots, n, l \in (L_{p_1}(\Gamma)/E_{p_1}(G))', \|l\| = 1\}, \\ n = 1, 2, \ldots, \end{aligned}$$

where $y_1, ..., y_n$ are continuous linear functionals on $E_a(G)$ and $\|l\|$ is the norm of the corresponding linear functional. Since the dual of $L_{p_1}(\Gamma)/E_{p_1}(G)$ is $E_b(G)$,

$$d^0(A_f) = \sup\left\{\left|\int_\Gamma (uvf)(\xi)\,d\xi\right| : u \in E_a(G),\, v \in E_b(G),\, \|u\|_a = \|v\|_b = 1\right\} \tag{16}$$

and

$$d^n(A_f) = \inf_{y_1, ..., y_n} \sup\left\{\left|\int_\Gamma (uvf)(\xi)\,d\xi\right| : u \in E_a(G),\, v \in E_b(G),\right.$$
$$\left.\|u\|_a = \|v\|_b = 1,\, y_i(u) = 0,\, i = 1, ..., n\right\}, \qquad n = 1, 2..., \tag{17}$$

where $y_1, ..., y_n$, are continuous linear functionals on $E_a(G)$. Here and in what follows we assume that $E_\infty(G)$ is given the topology of uniform convergence on compact subsets of G.

8. PROOF OF THEOREM 2.1

8.1. *Upper Bound for $l_n(A_f)$*

We show that

$$l_n(A_f) \leqslant \Delta_{n,p}, \qquad n = 0, 1, 2, ... \tag{18}$$

First, for an arbitrary function h in the class $\mathcal{M}_{n,p}(G)$ the Hankel operator A_h is a linear operator with rank at most n and, if $a = \infty$, is weak* continuous. Second, we have with the help of (14) and (15)

$$\|A_f - A_h\| = \|A_{f-h}\| \leqslant \|f - h\|_p.$$

Inequalities (18) follow immediately from the definitions of $l_n(A_f)$ and $\Delta_{n,p}$.

8.2. *Lower Bound for $d^n(A_f)$*

We now proceed to the proof of the inequalities

$$\Delta_{n,p} \leqslant d^n(A_f), \qquad n = 0, 1, 2, ... \tag{19}$$

It will be assumed that $n \geqslant 1$. The equality

$$\Delta_{0,p} = d^0(A_f)$$

follows immediately from (16), (11), the relation

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{q},$$

and the fact that any function $\varphi \in E_q(G)$, $\|\varphi\|_q = 1$, can be represented in the form $\varphi = uv$, where $u \in E_a(G)$, $v \in E_b(G)$, $\|u\|_a = 1$, $\|v\|_b = 1$.

In this subsection we confine ourselves to the case when f is holomorphic on Γ.

There exists an odd continuous mapping $\Phi\colon S^{2n+1} \to \mathscr{B}_n$, $\Phi(-\mathbf{w}) = -\Phi(\mathbf{w})$, from the sphere S^{2n+1} into the set of all Blaschke products of degree at most n in $E_\infty(G)$ when $E_\infty(G)$ is given the topology of uniform convergence on compact subsets of G. The mapping is defined as follows. Let $\xi_0, \xi_1, \ldots, \xi_n$ be $n+1$ distinct points in G. For each $(n+1)$-tuple $\mathbf{w} = (w_0, w_1, \ldots, w_n)$ of complex numbers with $\sum_{i=0}^n |w_i|^2 = 1$, we set

$$\Phi(\mathbf{w}) = B_{\mathbf{w}}(z),$$

where $B_{\mathbf{w}}(z)$ is a unique Blaschke product of degree at most n such that

$$\lambda B_{\mathbf{w}}(\xi_i) = w_i, \qquad i = 0, 1, \ldots, n,$$

and

$$\lambda = \inf\{\|F\|_\infty\colon F \in E_\infty(G),\ F(\xi_i) = w_i,\ i = 0, 1, \ldots, n\}$$

(see [4, 5] for more details).

For each Blaschke product B of degree at most n denote by φ_B the unique function such that $\varphi_B \in E_q(G)$, $\|\varphi_B\|_q = 1$, and

$$\int_\Gamma \varphi_B B f\, d\xi = \sup_{\tilde{\varphi} \in E_q(G),\ \|\tilde{\varphi}\|_q = 1} \left| \int_\Gamma (\tilde{\varphi} B f)(\xi)\, d\xi \right| \tag{20}$$

(see Lemma 5.1).

Fix $z_0 \in G$. Let us consider the case when $1 < a < \infty$. By (2) and (3), we get $1 < p < \infty$ and $1 < q < \infty$. Let

$$u_B(z) = \left(\frac{\varphi_B(z)}{\tilde{B}(z)}\right)^{q/a}, \qquad z \in G,$$

where $\tilde{B}(z)$ is a Blaschke product constructed by zeros of φ_B in G, $\varphi_B(z_0)/\tilde{B}(z_0) > 0$. Here and in what follows we take that branch of the

$(z)^{q/a}$, that is positive on the positive part of the real line. It is not hard to see that $u_B \in E_a(G)$,

$$u_B(\xi) \neq 0, \, \xi \in G, \qquad u_B(z_0) > 0, \tag{21}$$

and

$$|u_B(\xi)| = |\varphi_B(\xi)|^{q/a}$$

almost everywhere on Γ. If $a = \infty$, we set $u_B(\xi) \equiv 1$. Let

$$v_B(\xi) = \frac{\varphi_B(\xi)}{u_B(\xi)}, \qquad \xi \in G.$$

We remark that $v_B \in E_b(G)$ and $\|u_B\|_a = 1$, $\|v_B\|_b = 1$.

Let τ be the mapping from the sphere S^{2n+1} into $E_a(G)$ defined by

$$\tau(\mathbf{w}) = u_{\Phi(\mathbf{w})} \Phi(\mathbf{w}) = u_{B_\mathbf{w}} B_\mathbf{w},$$

where

$$\mathbf{w} = (w_0, \ldots, w_n) \in \mathbf{C}^{n+1}, \qquad \sum_{i=0}^{n} |w_i|^2 = 1.$$

Lemma 8.1. *The mapping τ is odd and continuous from S^{2n+1} into $E_a(G)$ when $E_a(G)$ is given the topology of uniform convergence on compact subsets of G.*

Proof. It will be assumed without loss of generality that $1 < a < \infty$. For $a = \infty$ the properties of the mapping τ follow directly from the corresponding properties of the mapping Φ.

Clearly $\tau(\mathbf{w})$ is odd. To prove the continuity, let $\mathbf{w}_k \to \mathbf{w}$ as $k \to \infty$. Let $\varphi_k = \varphi_{B_{\mathbf{w}_k}}$. Since $\|\varphi_k\|_q = 1$, $k = 0, 1, 2, \ldots$, it follows that there exist a sequence $\Lambda \subset \mathbf{N}$ and a function $g \in E_q(G)$, $\|g\|_q \leqslant 1$ such that $\varphi_k \to g$ uniformly on compact subsets of G as $k \to \infty$, $k \in \Lambda$. We show that $g = \varphi_{B_\mathbf{w}}$. Denote by γ any positively oriented contour such that $\gamma \subset G$ and f is holomorphic on γ. Let φ be an arbitrary function in $E_q(G)$, $\|\varphi\|_q = 1$. According to (20),

$$\int_\Gamma (\varphi_k B_{\mathbf{w}_k} f)(\xi) \, d\xi \geqslant \left| \int_\Gamma (\varphi B_{\mathbf{w}_k} f)(\xi) \, d\xi \right|, \qquad k = 0, 1, 2, \ldots.$$

Therefore,

$$\int_\gamma (\varphi_k B_{\mathbf{w}_k} f)(\xi) \, d\xi \geqslant \left| \int_\gamma (\varphi B_{\mathbf{w}_k} f)(\xi) \, d\xi \right|, \qquad k = 0, 1, 2, \ldots.$$

Since $B_{w_k} \to B_w$ uniformly on compact subsets of G as $k \to \infty$, we can pass to the limit as $k \to \infty$, $k \in \Lambda$, obtaining

$$\int_\gamma (gB_w f)(\xi)\, d\xi \geqslant \left| \int_\gamma (\varphi B_w f)(\xi)\, d\xi \right|$$

and

$$\int_\Gamma (gB_w f)(\xi)\, d\xi \geqslant \left| \int_\Gamma (\varphi B_w f)(\xi)\, d\xi \right|.$$

Then, by Lemma 5.1, we obtain that

$$g = \varphi_{B_w},$$

and $\varphi_k \to \varphi_{B_w}$ uniformly on compact subsets of G as $k \to \infty$.

We now prove that $u_{B_{w_k}} \to u_{B_w}$ uniformly on compact subsets of G as $k \to \infty$ (cf. [3, Lemma 3]). Let $u_k = u_{B_{w_k}}$, and $v_k = v_{B_{w_k}}$. Since $\|u_k\|_a = 1$ and $\|v_k\|_b = 1$, it follows that there exist a sequence $\Lambda_1 \subset \mathbf{N}$, functions $u \in E_a(G)$ and $v \in E_b(G)$ such that $u_k \to u$ and $v_k \to v$ uniformly on compact subsets of G as $k \to \infty$, $k \in \Lambda_1$. By (21), $u(z_0) > 0$ and $u(\xi) \neq 0$, $\xi \in G$. We have

$$\|u\|_a \leqslant \lim_{k\to\infty} \|u_k\|_a = 1 \qquad \text{and} \qquad \|v\|_b \leqslant \lim_{k\to\infty} \|v_k\|_b = 1.$$

Using the equalities $\|\varphi_{B_w}\|_q = 1$ and $\varphi_{B_w} = uv$, we can write

$$1 = \|\varphi_{B_w}\|_q \leqslant \|u\|_a \cdot \|v\|_b \leqslant 1. \tag{22}$$

Relation (22) implies that

$$\|u\|_a = \|v\|_b = 1,$$
$$|u(\xi)|^a = |v(\xi)|^b$$

almost everywhere on Γ and

$$|u(\xi)| = |\varphi_{B_w}(\xi)|^{q/a}$$

almost everywhere on Γ. Since f is holomorphic on Γ, it follows that φ_{B_w}, has no a nonconstant singular factor (see Lemma 5.1). From this, the formula $\varphi_{B_w} = u_{B_w} v_{B_w}$ and the result that factors in the factorization theorem (see Section 4) are unique up to multiplication by unimodular scalars, we get that u_{B_w} has no a nonconstant singular inner factor. Analogously, we can conclude that u has no a nonconstant singular inner factor. Hence, u_B and u are outer functions. Since $|u_{B_w}(\xi)| = |u(\xi)|$ almost everywhere on Γ, and $u_{B_{\mathbf{w}}}(z_0) > 0$, $u(z_0) > 0$, it follows from this that $u = u_{B_{\mathbf{w}}}$. Thus, $u_{B_{\mathbf{w}}} \to u_{B_{\mathbf{w}}}$ uniformly on compact subsets of G as $k \to \infty$. Therefore, $u_{B_{\mathbf{w}_k}} B_{\mathbf{w}_k} \to u_{B_{\mathbf{w}}} B_{\mathbf{w}}$, uniformly on compact subsets of G as $k \to \infty$. This shows that the mapping τ is continuous. ∎

Before continuing with the proof of the theorem, we formulate an auxiliary result. The following Borsuk's theorem is valid (see [9]).

Let Ψ be an odd continuous mapping from the sphere $S^k \subseteq \mathbf{R}^{k+1}$ into $\mathbf{R}^k$. Then there exists a point $x_0 \in S^k$ such that $\Psi(x_0) = 0$.

We now apply standard arguments involving Borsuk's theorem to prove (19). Let $y_1, \ldots, y_n$ be n continuous linear functionals on $E_a(G)$. The mapping

$$\Phi(\mathbf{w}) = (y_1(\tau(\mathbf{w})), \ldots, y_n(\tau(\mathbf{w}))), \qquad \mathbf{w} = (w_0, \ldots, w_n),$$

is continuous and odd from S^{2n+1} into $\mathbf{C}^n$. It follows from Borsuk's theorem that this map has a zero; that is, that there is $B^* \in \mathscr{B}_n$ such that

$$y_i(u_{B^*} B^*) = 0, \qquad i = 1, \ldots, n.$$

Hence,

$$\begin{aligned}
&\sup \left\{ \left| \int_\Gamma (uvf)(\xi)\, d\xi \right| : u \in E_a(G),\, v \in E_b(G),\, \|u\|_a = \|v\|_b = 1, \right.\\
&\qquad \left. y_i(u) = 0,\, i = 1, \ldots, n \right\}\\
&\geqslant \left| \int_\Gamma (u_{B^*} v_{B^*} B^* f)(\xi)\, d\xi \right| = \left| \int_\Gamma (\varphi_{B^*} B^* f)(\xi)\, d\xi \right|\\
&= \sup_{\tilde\varphi \in E_q(G),\, \|\tilde\varphi\|_q = 1} \left| \int_\Gamma (\tilde\varphi B^* f)(\xi)\, d\xi \right|\\
&\geqslant \inf_{B \in \mathscr{B}_n} \sup_{\tilde\varphi \in E_q(G),\, \|\tilde\varphi\|_q = 1} \left| \int_\Gamma (\tilde\varphi B f)(\xi)\, d\xi \right| = \Delta_{n,p}.
\end{aligned}$$

When we minimize over all choices of $y_1, \ldots, y_n$, we obtain (see (17)) the desired lower bound (19) for $d^n(A_f)$ and then the equality $d^n(A_f) = \Delta_{n,p}$.

8.3. $\Delta_{n,p} = d^n(A_f)$. *The Case $f \in L_p(\Gamma)$*

Let $f \in L_p(\Gamma)$. Fix an integer $n \geqslant 1$. Let us consider a sequence $\{r_\nu\}$, $\nu = 0, 1, 2, \ldots$, of rational functions with poles off Γ such that $\|f - r_\nu\|_p \to 0$ as $\nu \to \infty$. Fix an integer $\nu \geqslant 0$. With the help of (8) and (14) we can write the inequalities

$$|d^n(A_f) - d^n(A_{r_\nu})| \leqslant \|A_f - A_{r_\nu}\| \leqslant \|f - r_\nu\|_p. \tag{23}$$

We can see directly from the definition of $\Delta_{n,p}$ that

$$|\Delta_{n,p}(f; G) - \Delta_{n,p}(r_\nu; G)| \leqslant \|f - r_\nu\|_p. \tag{24}$$

Using now the fact that the function r_ν is holomorphic on Γ, we obtain the following equality

$$\Delta_{n,p}(r_\nu; G) = d^n(A_{r_\nu}).$$

From this, passing to the limit as $\nu \to +\infty$, by (23) and (24), we get

$$\Delta_{n,p}(f; G) = d^n(A_f).$$

Theorem 2.1 is proved.

9. PROOF OF COROLLARY 2.1

We first show

$$A'_f = A_f, \tag{25}$$

where A'_f is the dual of the Hankel operator A_f. Indeed, since

$$\frac{1}{p_1} + \frac{1}{2q} = 1$$

and the dual of $E_{2q}(G)$ is $L_{p_1}(\Gamma)/E_{p_1}(G)$ and $L_{p_1}(\Gamma)/E_{p_1}(G)$ is the dual of $E_{2q}(G)$, we obtain that the dual operator A'_f is an operator which maps $E_{2q}(G)$ into $L_{p_1}(\Gamma)/E_{p_1}(G)$. Moreover,

$$\langle A_f \alpha, v \rangle = \int_\Gamma (\alpha f)(\xi)\, v(\xi)\, d\xi = \langle \alpha, A'_f v \rangle$$

for all $\alpha, v \in E_{2q}(G)$. It follows from this that the dual operator $A'_f \colon E_{2q}(G) \to L_{p_1}(\Gamma)/E_{p_1}(G)$ of A_f is the composition of the operator of multiplication by the function f and the canonical surjection $\mathbf{P}$ from $L_{p_1}(\Gamma)$ onto the quotient space $L_{p_1}(\Gamma)/E_{p_1}(G)$. Hence $A'_f = A_f$. From this, on the basis of Theorem 2.1 and (10), we get (7).

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